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Existence and Uniqueness of Positive Solutions to a Linear Transport Equation in a Metric Space

A. Majorana
Dipartimento di Matematica, Università di Catania
Viale A. Doria 6, 95125 Catania, Italy
majorana@dipmat.unict.it

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Abstract—The linear functional equation $\partial_t z = L(z) - rz$ is considered. The linear operator L acts on a linear metric space of real functions z depending on t and on a parameter w belonging to a subset of $\mathbb{R}^m$. The existence and uniqueness to a nonnegative solution of the initial value problem is shown. An application to a kinetic equation is performed.

Keywords—Transport equations, Initial value problem, Continuous solutions.

1. INTRODUCTION AND MAIN RESULT

In this paper, we prove an existence and uniqueness theorem for a linear functional equation. The proof is achieved using simple well-known techniques based on the Banach Fixed-Point Theorem and on the properties of monotone operators. The usual frameworks for such operators are the semiordered or normed spaces (see, for instance, [1,2]), where additional properties, as the existence of a least upper bound for bounded subsets or the completeness with respect to some norm, are assumed. We study the problem in suitable metric spaces, so that our result requires only weak assumptions. Nevertheless, these insure good properties of the solution. Since the simple structure of the linear metric space, where we look for the solution, we think that the result below is not an easy application of known theorems.

We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_0^+$ the set of positive integer, of the real numbers, and nonnegative real numbers, respectively.

Let $W \subseteq \mathbb{R}^m$ ($m \in \mathbb{N}$) be a nonempty closed set. We consider two linear subsets of the space F of all real functions $\phi : \mathbb{R}_0^+ \times W \to \mathbb{R}$. The first, which is denoted by X, consists of all functions $\phi \in F$ such that

(i) for every $n \in \mathbb{N}$, the set $\{|\phi(t,w)| : (t,w) \in [0,n] \times W\}$ is bounded;
(ii) for any $w \in W$, the function $t \to \phi(t,w)$ is continuous in $\mathbb{R}_0^+$.

The linear space $Y = X \bigcap C^0(\mathbb{R}_0^+ \times W)$ is the other subset of F.

In the following, we indicate with S the linear space X or, in the alternative, Y. We denote by S_0 the subset of S containing all nonnegative functions, which are constant with respect to t.

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We consider a linear operator $L : S \to S$ satisfying the following hypotheses.

HYPOTHESIS (L_1). *If ϕ and ψ belong to S,*

$$\phi(t,w) \le \psi(t,w), \qquad \text{for every } (t,w) \in [0,n] \times W,$$

implies

$$L(\phi)(t,w) \le L(\psi)(t,w), \qquad \text{for every } (t,w) \in [0,n] \times W.$$

HYPOTHESIS (L_2). *There exists a function $b \in S_0$ such that for any $\varphi \in S$, but constant with respect to w, there follows*

$$|L(\varphi)(t,w)| \le b(w)|\varphi(t)|, \qquad \text{for every } (t,w) \in \mathbb{R}_0^+ \times W.$$

As a consequence of the first assumption and the linearity of L, we note that, for any $\phi \in S$, we have

$$|L(\phi)(t,w)| \le L(|\phi|)(t,w), \qquad \text{for every } (t,w) \in \mathbb{R}_0^+ \times W.$$

Let $r \in S_0$ be assigned. We consider the functional equation

$$\frac{\partial z}{\partial t} = L(z) - rz, \tag{1}$$

with the initial condition

$$z(0,w) = z_0(w), \qquad \text{for every } w \in W. \tag{2}$$

For solution of problem (1),(2) we intend a function z belonging to S and having continuous partial derivative with respect to t.

The linear space S can be equipped by a metric in the following way. Let

$$\lambda = \sup\left\{2b(w) - r(w) : w \in W\right\}. \tag{3}$$

We introduce a countable separating family of seminorms on S, by setting, for every $n \in \mathbb{N}$ and for each $\phi \in S$,

$$p_n\phi \overset{\text{def}}{=} \sup\left\{e^{-\lambda t}|\phi(t,w)| : (t,w) \in [0,n] \times W\right\}. \tag{4}$$

Then a metric can be defined by means of the following distance:

$$d(\phi,\psi) \overset{\text{def}}{=} \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(\phi-\psi)}{1+p_n(\phi-\psi)}. \tag{5}$$

It is straightforward to verify that the space S is complete.

THEOREM 1. *Let L be a linear operator satisfying assumptions (L_1),(L_2). If r and z_0 belong to S_0, then the Cauchy problem (1),(2) admits an unique nonnegative solution belonging to S.*

PROOF. It is evident that $z : \mathbb{R}_0^+ \times W \to \mathbb{R}$ is a solution of equations (1),(2) if and only if it is a solution of the integral equation

$$z(t,w) = z_0(w)e^{-r(w)t} + \int_0^t e^{-r(w)(t-s)} L(z)(s,w)\,ds. \tag{6}$$

We denote by $T(z)(t,w)$ the right-hand side of equation (6). It is easy to check that T is an operator, which maps any $z \in S$ in a function belonging to S. Now, we prove that T is a contraction on S with constant $1/2$. In fact, fix $n \in \mathbb{N}$, for any $(t,w) \in [0,n] \times W$, we have

$$\begin{aligned}
\left|\int_0^t e^{-r(w)(t-s)} L(\vartheta)(s,w)\,ds\right| &\le \int_0^t e^{-r(w)(t-s)} L(|\vartheta|)(s,w)\,ds \\
\le \int_0^t e^{-r(w)(t-s)} b(w) e^{\lambda s} p_n\vartheta\,ds &\le \frac{1}{2}\left[r(w)+\lambda\right] p_n\vartheta e^{-r(w)t} \int_0^t e^{[r(w)+\lambda]s}\,ds \\
&= \frac{1}{2} p_n\vartheta e^{-r(w)t}\left\{e^{[r(w)+\lambda]t} - 1\right\} \le \frac{1}{2} p_n(\vartheta) e^{\lambda t},
\end{aligned}$$

for any $\vartheta \in S$.

Here, we have used the inequality $|\vartheta(s,w)| \leq p_n \vartheta e^{\lambda s}$, for every $(s,w) \in [0,n] \times W$ and the property (L_2), being the function $(s,w) \to p_n \phi e^{\lambda s}$ an element of S. Therefore, by taking into account the linearity of L, we obtain

$$p_n(T(\phi) - T(\psi)) \leq \frac{1}{2} p_n(\phi - \psi), \qquad \text{for all } \phi, \psi \in S.$$

This implies that

$$d(T(\phi), T(\psi)) \leq \frac{1}{2} d(\phi, \psi), \qquad \text{for all } \phi, \psi \in S.$$

Hence, the Banach Fixed-Point Theorem insures the existence of a unique solution of equation (6). Since this can be obtained by iteration, if we consider the sequence defined by

$$\phi_0(t,w) = 0, \quad \phi_n = T(\phi_{n-1}), \qquad \text{for every } n \in \mathbb{N}, \tag{7}$$

then ϕ_n converges to the solution. It is easy to show that the above sequence is nondecreasing. In fact, we have $0 = \phi_0(t,w) \leq \phi_1(t,w) = z_0(w)e^{-r(w)t}$, and, by induction, we obtain

$$\phi_{n+1}(t,w) - \phi_n(t,w) = \int_0^t e^{-r(w)(t-s)} L(\phi_n - \phi_{n-1})(s,w)\, ds \geq 0.$$

This proves that the solution z is nonnegative and completes the proof. ∎

A simple upper bound for the solution can be obtained, using the sequence (7). Let

$$\mu = \sup \{b(w) - r(w) : w \in W\}.$$

It is simple to prove by induction that

$$z(t,w) \leq \sup \{z_0(w) : w \in W\} e^{\mu t}, \qquad \text{for every } (t,w) \in \mathbb{R}_0^+ \times W.$$

Under our hypotheses, it represents the best upper bound. For instance, this follows, considering the trivial case $L(z) = bz$, with b constant, and $r = 0$.

2. AN APPLICATION TO A KINETIC EQUATION

The evolution problem for kinetic transport [3] of particles occupying the whole space $\mathbb{R}^3$ without a source term is written

$$\frac{\partial f}{\partial t}(t,x,\xi) + \xi \cdot \frac{\partial f}{\partial x}(t,x,\xi) = \int_V K(\xi,\xi_*) f(t,x,\xi_*)\, d\xi_* - \nu(\xi) f(t,x,\xi), \tag{8}$$

$$f(0,x,\xi) = f_0(x,\xi), \tag{9}$$

where $x \in \mathbb{R}^3$ and $\xi \in V$ (a nonempty closed subset of $\mathbb{R}^3$). The unknown f is the probability density to find a particle at time t and position x with velocity ξ. The kernel K and the collision frequency ν are assigned nonnegative functions, which usually depend on the physical characteristics of the host medium.

In mild form (see, for instance, [4–6]), equations (8),(9) can be written as

$$\frac{\partial f^\#}{\partial t}(t,x,\xi) = \int_V K(\xi,\xi_*) f^\# [t, x + t(\xi - \xi_*), \xi_*]\, d\xi_* - \nu(\xi) f^\#(t,x,\xi), \tag{10}$$

$$f^\#(0,x,\xi) = f_0(x,\xi), \tag{11}$$

where $f^\#(t,x,\xi) = f(t, x + t\xi, \xi)$.

We look for a nonnegative solution of the initial value problem (10),(11) in the vector space M of the functions $f^{\#} \in C^0(\mathbb{R}_0^+ \times \mathbb{R}^3 \times V)$, such that

$$s_n(f^{\#}) \stackrel{\text{def}}{=} \sup\left\{|f^{\#}(t,x,\xi)| : (t,x,\xi) \in [0,n] \times \mathbb{R}^3 \times V\right\} < +\infty,$$

for every $n \in \mathbb{N}$. This space coincides with Y, choosing $W = \mathbb{R}^3 \times V$.

We assume the following.

(a_1) For every $\xi \in V$, the nonnegative function $\xi_* \to K(\xi,\xi_*)$ belongs to $L^1(V)$.

(a_2) The function $\xi \to \beta(\xi) = \int_V K(\xi,\xi_*)\,d\xi_*$ is bounded in V.

(a_3) Let $V_n = \{\xi \in V : \|\xi\| \le n\}$ ($\|\cdot\|$ indicates the Euclidean norm in $\mathbb{R}^3$), for every $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \sup\left\{\int_{V-V_n} K(\xi,\xi_*)\,d\xi_* : \xi \in V\right\} = 0.$$

(a_4) If η is a limit point of V, then

$$\lim_{\xi\to\eta} \int_V |K(\xi,\xi_*) - K(\eta,\xi_*)|\,d\xi_* = 0.$$

(a_5) $\nu : V \to \mathbb{R}_0^+$ is continuous and bounded.

Now, the role of L is assumed by the integral operator in (10). We denote it by $L(f^{\#})$. We first prove that L maps M into itself. We have

$$s_n(L(f^{\#})) \le s_n(f^{\#}) \sup_{\xi\in V} \beta(\xi), \quad \text{for every } n \in \mathbb{N}$$

and boundness is achieved by (a_2). We next demonstrate that $L(f^{\#})$ is continuous. Let be η a limit point of V, $\tau \in \mathbb{R}_0^+$ and $y \in \mathbb{R}^3$. We choose an integer $l > \tau$ and a positive number $\delta \le l - \tau$. Let us consider the set

$$\Delta = \left\{(t,x,\xi) \in [0,l] \times \mathbb{R}^3 \times V : |t-\tau| + \|x-y\| + \|\xi-\eta\| \le \delta\right\}.$$

Now, taking into account that we can write $L(f^{\#})$ as

$$L(f^{\#})(t,x,\xi) = \int_V K(\xi,\xi_*) f(t, x+t\xi, \xi_*)\,d\xi_*,$$

for every $(t,x,\xi) \in \Delta$, we have

$$\begin{aligned}
&\left|L(f^{\#})(t,x,\xi) - L(f^{\#})(\tau,y,\eta)\right| \\
&\quad \le \int_V K(\xi,\xi_*)\,|f(t,x+t\xi,\xi_*) - f(\tau,y+\tau\eta,\xi_*)|\,d\xi_* \\
&\qquad + \int_V |f(\tau,y+\tau\eta,\xi_*)|\,|K(\xi,\xi_*) - K(\eta,\xi_*)|\,d\xi_* \\
&\quad \le \int_{V_n} K(\xi,\xi_*)\,|f(t,x+t\xi,\xi_*) - f(\tau,y+\tau\eta,\xi_*)|\,d\xi_* \\
&\qquad + \int_{V-V_n} K(\xi,\xi_*)\,|f(t,x+t\xi,\xi_*) - f(\tau,y+\tau\eta,\xi_*)|\,d\xi_* \\
&\qquad + s_l\left(f^{\#}\right) \int_V |K(\xi,\xi_*) - K(\eta,\xi_*)|\,d\xi_* \\
&\quad \le \sup\{|f(t,x+t\xi,\xi_*) - f(\tau,y+\tau\eta,\xi_*)| : (t,x,\xi,\xi_*) \in \Delta \times V_n\} \int_{V_n} K(\xi,\xi_*)\,d\xi_* \\
&\qquad + s_l\left(f^{\#}\right)\left[2\int_{V-V_n} K(\xi,\xi_*)\,d\xi_* + \int_V |K(\xi,\xi_*) - K(\eta,\xi_*)|\,d\xi_*\right].
\end{aligned}$$

Since the function $(t,x,\xi,\xi_*) \to f(t,x+t\xi,\xi_*)$ is uniformly continuous on the compact set $\Delta \times V_n$, and (a_2)–(a_4) hold, the proof is accomplished. Moreover, it is a simple matter to show that conditions (L_1) and (L_2) are satisfied. Thus, applying Theorem 1, we deduce that, if $f_0 \in C^0(\mathbb{R}^3 \times V)$ is nonnegative, then the Cauchy problem (10),(11) admits an unique nonnegative solution belonging to M.

We note that assumptions (a_2)–(a_4) are immediately verified if V is a compact set and the kernel K is continuous. Otherwise, in general, the treatment of integral operators in unbounded domains and in the framework of C^0 spaces, requires, as the previous application, additional hypotheses (see, for instance, [7,8]).

If a condition of integrability with respect to the variable ξ for the function f is required, then a simple change of the unknown, as $g(t,x,\xi) = (1+\|\xi\|^\alpha)\, f(t,x,\xi)$, for a suitable choice of the parameter α, could guarantee the result. This essentially depends on the kernel K.

Another remark concerns the applicability of Theorem 1. In the above application, the results have been obtained without verifying or assuming, for instance, the continuity or the compactness of the operator L in the metric space M.

3. CONCLUDING REMARKS

Our intent has been to exhibit a simple framework, where linear transport equation can be studied. We have applied this technique to prove the existence of solutions to the Boltzmann equation, which describes the evolution of an electron gas in a semiconductor [9]. Some extensions are quite evident. For example, to add a source term in the linear transport equation.

Other situations can be considered. We list some of them, which are objects of present or future research. The case of only locally bounded solutions with respect to the parameter w may be of interest in order to extend the applicability of Theorem 1. The possibility of studying a finite-dimensional vectorial transport equation is another goal. Finally, we hope to extend the technique to nonlinear transport equations, where a Lipschitz-type condition is assumed on the operator. Partial results on these topics are already obtained.

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